# The weak orthogonality between generalized Möbius functions and bounded sequences 

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#### Abstract

In this paper, we will show that the generalized Möbius functions associated to a certain family of $L$-functions are weakly orthogonal to bounded sequences.


## 1 Introduction

The well-known Prime Number Theorem has several equivalent forms, one of which saying that

$$
\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n)=0
$$

where $\mu(n)$ denotes the Möbius function, defined by the following Dirichlet series:

$$
\zeta(s)^{-1}=\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}}
$$

for $\Re(s)>1$, with $\zeta(s)$ being the Riemann zeta function. This result can be rephrased in the following general framework: Let $a(n)$ and $b(n)$ be two arithmetic functions. We say they are asymptotically orthogonal to each other, if

$$
\lim _{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^{N} a(n) b(n)=0
$$

We say they are weakly orthogonal to each other, if there exists some $\delta>0$ such that

$$
\sum_{n=1}^{N} a(n) b(n)=O\left(\frac{N}{\log ^{\delta} N}\right)
$$

We say they are strongly orthogonal to each other, if for any (large) $A>0$, we have

$$
\sum_{n=1}^{N} a(n) b(n)=O_{A}\left(\frac{N}{\log ^{A} N}\right)
$$

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where the notation $O_{A}$ means the implied constant depends on the choice of $A$. Obviously, strong orthogonality implies weak orthogonality, and the later implies asymptotic orthogonality. Using this language, the Prime Number Theorem mentioned above can be restated as: The Möbius function $\mu(n)$ is asymptotically orthogonal to the constant function 1.

Davenport's work [Dav37] expanded the above result. He showed that the Möbius function $\mu(n)$ is strongly orthogonal to any linear phrase $e(n \alpha)$, where $\alpha$ is any real number. Moreover, the estimate is uniform in $\alpha$. It should be noted that a key step in Davenport's proof is to make use of Siegel's Theorem, which states that for any $\epsilon>0$, there exists an (ineffective) constant $c_{\epsilon}>0$ such that for any non-principal Dirichlet character $\chi$ with conductor $q$, the Dirichlet $L$-function $L(s, \chi)$ has no zero in the interval

$$
\left(1-\frac{c_{\epsilon}}{q^{\epsilon}}, 1\right)
$$

In 2012, Green and Tao [GT12] showed that $\mu(n)$ is strongly orthogonal to any nilsequence. Later it was observed and conjectured by Sarnak [Sar12], [Sar] that the Möbius function $\mu(n)$ behaves so randomly that it should be asymptotically orthogonal to any "low-complexity" sequence. The term "low-complexity" can be made precise using dynamical language. Explicitly, let ( $X, T$ ) be a flow with zero entropy. Then for any point $x \in X$ and any continuous function $f: X \rightarrow \mathbb{C}$, Sarnak's conjecture predicts that the arithmetic function $f\left(T^{n} x\right)$ should be asymptotically orthogonal to $\mu(n)$.

In the past ten years after Sarnak's conjecture was formally proposed, several special cases have been verified, leading to applications to many branches of mathematics including number theory, ergodic theory and others. To mention an example, Liu and Sarnak [LS15] verified the case where $(X, T)$ is an affine linear flow on a compact abelian group with zero entropy.

Sarnak's conjecture can be naturally generalized in view of the following analytic identity

$$
\zeta(s)^{-1}=\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s}}
$$

If we replace the Riemann zeta function $\zeta(s)$ by any $L$-function $L(s, \pi)$ of degree $d+1$ with Euler product expansion

$$
L(s, \pi)=\prod_{p} \prod_{i=1}^{d+1}\left(1-\frac{\alpha_{i}(p)}{p^{s}}\right)^{-1}
$$

for $\Re(s)>1$, then its reciprocal is a Dirichlet series. We denote the coefficients of that series by $\mu_{\pi}(n)$ and call it the Möbius function associated with $\pi$. Thus the following analogous identity holds:

$$
L(s, \pi)^{-1}=\sum_{n=1}^{\infty} \frac{\mu_{\pi}(n)}{n^{s}}
$$

for $\Re(s)>1$. The arithmetic function $\mu_{\pi}(n)$ shares similar properties with the usual Möbius function $\mu(n)$. For example, it is multiplicative and is supported on $(d+2)^{t h}$ power free integers.

It then can be asked if the generalized Möbius function $\mu_{\pi}(n)$ is asymptotically (or weakly/strongly) orthogonal to any "low-complexity" sequence. Jiang and Lü [JL19] studied this question when $\pi$ is a Maass form for $S L(2, \mathbb{Z})$ or $S L(3, \mathbb{Z})$. Explicitly, they proved that if $F$ is a Maass form for
$S L(2, \mathbb{Z})$ or $S L(3, \mathbb{Z})$, then there exists a constant $c_{F}>0$ such that for any real number $\alpha$, we have

$$
\sum_{n=1}^{N} \mu_{F}(n) e(n \alpha)=O\left(N e^{-c_{F} \sqrt{\log N}}\right)
$$

where the implied constant does not depend on $\alpha$. In particular, this implies $\mu_{F}(n)$ is strongly orthogonal to any linear phrase $e(n \alpha)$, a natural generalization of Davenport's work [Dav37]. It should be noted that, like in Davenport's proof, a key step in Jiang and Lv's proof is Banks' result [Ban97] (resp. Hoffstein and Ramakrishnan's result [HR95]) that $G L(3)$ (resp. $G L(2)$ ) cusp forms do not admit Siegel zeros. Results of this power allow them to get much better estimate than just arbitrary logarithm power saving as in Davenport's result.

It seems natural to employ results about non-existence of Siegel zeros to study orthogonality between generalized Möbius functions and linear phrases. But unfortunately very limited knowledge about Siegel zeros is known for higher rank cases (or higher degree cases). In this paper, we will take a different approach to show:

$$
\sum_{n=1}^{N}\left|\mu_{\pi}(n)\right|=O\left(\frac{N}{\log ^{\delta} N}\right)
$$

for certain $L$-functions $L(s, \pi)$. (Here $\delta$ is some positive number depending only on the degree of $L(s, \pi)$.)

Theorem 1.1. Let $d \geq 1$ be a positive integer. Let $L(s, \pi)$ be a self-dual, everywhere unramified $L$-function of degree $d+1$. Suppose that $L(s, \pi)$ satisfies the Ramanujan conjecture and for $x>0$,
(a) $\sum_{p \leq x} \frac{\lambda_{\pi}(p)^{2}}{p}=\log \log x+O(1) ;$
(b) $\sum_{p \leq x} \frac{\lambda_{\pi}(p)^{4}}{p} \geq(d+1) \log \log x+O(1)$.

Then we have, for $x>0$,

$$
\sum_{n \leq x}\left|\mu_{\pi}(n)\right| \ll \frac{x}{(\log x)^{\delta_{d}}}
$$

for some $\delta_{d}>0$. Here $\delta_{d}$ will be defined in Equation (1).
As a direct corollary, Theorem 1.1 implies that $\mu_{\pi}(n)$ is weakly orthogonal to any bounded sequences:

Corollary 1.2. Let $L(s, \pi)$ be an L-function as in Theorem 1.1. Let $\{b(n)\}$ be a bounded sequence. Then for $x>0$, we have

$$
\lim _{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \mu_{\pi}(n) b(n)=0
$$

This implies that $\mu_{\pi}(n)$ is weakly orthogonal to any "low-complexity" sequence described earlier, since such sequences are bounded.

Corollary 1.3. Let $L(s, \pi)$ be an L-function as in Theorem 1.1. Let $X$ be a compact space and $T: X \rightarrow X$ be a continuous map of zero entropy. Then for $x>0$, any continuous function $f: X \rightarrow \mathbb{C}$ and $x_{0} \in X$, we have

$$
\lim _{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} \mu_{\pi}(n) f\left(T^{n} x_{0}\right)=0
$$

Here the convergence is only dependent on the sup-norm of $f$.
We then give a family of examples satisfying the conditions in Theorem 1.1-the Rankin-Selberg $L$-functions

$$
L\left(s, \operatorname{Sym}^{m_{1}} f \otimes \operatorname{Sym}^{m_{2}} g\right)
$$

where $f, g$ are distinct normalized holomorphic Hecke eigenforms for $S L(2, \mathbb{Z})$. The result is formulated as follows:

Corollary 1.4. Let $f, g$ be distinct normalized holomorphic Hecke eigenforms for $\mathrm{SL}_{2}(\mathbb{Z})$. Then for any positive integers $m_{1}, m_{2} \geq 1$, we have:

$$
\sum_{n \leq x}\left|\mu_{\pi}(n)\right| \ll \frac{x}{(\log x)^{\delta}}
$$

where $\pi=\operatorname{Sym}^{m_{1}} f \otimes \operatorname{Sym}^{m_{2}} g$ and $\delta=\delta_{\left(m_{1}+1\right)\left(m_{2}+1\right)-1}$.
This corollary relies on the functoriality result by James Newton and Jack Thorne [NT21] showing that for a holomorphic cusp form $f$ of $\mathrm{SL}_{2}(\mathbb{Z})$, its symmetric power lifting $\operatorname{Sym}^{m} f$ is an automorphic cuspidal representation of $\mathrm{GL}_{m+1}$.

Finally, we establish the following theorem for Maass forms of $S L(2, \mathbb{Z})$ :
Theorem 1.5. Let $\phi$ be a normalized Hecke Maass form for $\mathrm{SL}_{2}(\mathbb{Z})$. Then for any $x>0$,

$$
\sum_{n \leq x}\left|\mu_{\phi}(n)\right| \ll \frac{x}{(\log x)^{1 / 12}} .
$$

This can be much harder since we do not have the Ramanujan-type bound nor the full functoriality result. However, we can still follow the idea of Theorem 1.1 and Corollary 1.4 to prove Theorem 1.5. As a direct application, this also implies that $\mu_{\phi}(n)$ is weakly orthogonal to any bounded sequences and hence "low-complexity" sequences.

## 2 Proof of Theorem 1.1

Proof of Theorem 1.1 Since $\pi$ satisfies the Ramanujan conjecture, we can apply Shiu's result [Shi80, Theorem 1] on the sums of nonnegative multiplicative functions. That implies:

$$
\sum_{n \leq x}\left|\mu_{\pi}(n)\right| \ll \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{\left|\mu_{\pi}(p)\right|}{p}\right)
$$

Notice that $\mu_{\pi}(p)=-\lambda_{\pi}(p)$. It suffices to show that

$$
\sum_{p \leq x} \frac{\left|\lambda_{\pi}(p)\right|}{p} \leq\left(1-\delta_{d}\right) \log \log x .
$$

The left proof used the idea of [TW16, Thereom 4] in Section 3.2. We need to replace all $m$ by $d$ in [TW16, Lemma 3.3]. Notice that in their lemma, one has $a_{2}(m)<0$ and hence it suffices to show that

$$
\sum_{p \leq x} \frac{\lambda_{\pi}(p)^{4}}{p} \geq(d+1) \log \log x+O(1)
$$

It was shown in [TW16, Lemma 3.3] that

$$
\begin{equation*}
\delta_{d}=\frac{d(d+2)}{d^{2}+3 d+1}\left(\frac{d+3}{d+2}-\sqrt{\frac{d+2}{d+1}}\right)>0 . \tag{1}
\end{equation*}
$$

## 3 Proof of Corollary 1.4

Before the proof of the corollary, we need several lemmas. By the functionality lifting, we have
Lemma 3.1. Let $f$ be a normalized holomorphic Hecke eigenforms for $\mathrm{SL}_{2}(\mathbb{Z})$. Then for positive integers $m_{1} \geq m_{2} \geq 1$, we have:

$$
\operatorname{Sym}^{m_{1}} f \otimes \operatorname{Sym}^{m_{2}} f \simeq \operatorname{Sym}^{m_{1}+m_{2}} f \boxplus \operatorname{Sym}^{m_{1}+m_{2}-2} f \cdots \operatorname{Sym}^{m_{1}-m_{2}+2} f \boxplus \operatorname{Sym}^{m_{1}-m_{2}} f .
$$

The last term is the trivial representation when $m_{1}=m_{2}$.
For the second lemma, we need several density functions on the set of primes: denote by $\mathcal{P}$ the set of all prime numbers and let $S$ be a subset of $\mathcal{P}$. Denote by $\bar{\delta}(S)$ (resp. $\underline{\delta}(S)$ ) the upper (resp. lower) Dirichlet density of $S$. Denote by $\bar{d}(S)$ (resp. $\underline{d}(S)$ ) the upper (resp. lower) natural density of $S$. It can be shown that, for any set $S \subseteq \mathcal{P}$,

$$
0 \leq \underline{d}(S) \leq \underline{\delta}(S) \leq \bar{\delta}(S) \leq \bar{d}(S) \leq 1
$$

Then we can prove the following lemma:
Lemma 3.2. Let $f, g$ be distinct normalized holomorphic Hecke eigenforms for $\mathrm{SL}_{2}(\mathbb{Z})$. Then for any positive integer $m \geq 1, \operatorname{Sym}^{m} f$ is not isomorphic to $\mathrm{Sym}^{m} g$ as cuspidal representations on $\mathrm{GL}_{m+1}$. Moreover, for any positive integers $m_{1}, m_{2} \geq 1 \operatorname{Sym}^{m_{1}} f$ is not isomorphic to $\operatorname{Sym}^{m_{2}} g$.

Proof It suffices to consider the case $m_{1}=m_{2}=m$. For $m=2$, this is true because they are eigenforms on $\mathrm{SL}_{2}(\mathbb{Z})$ and one cannot be the quadratic twist of the other one [Ram00]. So we would focus in the case $m \geq 3$. This follows the idea of [CM04, Proposition 5.1].

We first consider the case when $m$ is odd. Set:

$$
S=\left\{p \in \mathcal{P} \mid a_{f}(p) \neq a_{g}(p)\right\} .
$$

Then we know that $\underline{\delta}(S) \geq 1 / 4$ by [Wal14, Theorem 1].
Fix $m \geq 3$. Assume that $\operatorname{Sym}^{m} f$ is isomorphic to $\operatorname{Sym}^{m} g$. Denote by $\left\{\alpha_{p}, \alpha_{p}^{-1}\right\}$ (resp. $\left\{\beta_{p}, \beta_{p}^{-1}\right\}$ ) the Satake parameters of $f$ (resp. $g$ ) at the prime $p$. Then for all $p$, we have

$$
\left\{\alpha_{p}^{m}, \alpha_{p}^{m-2}, \ldots, \alpha_{p}^{2-m}, \alpha_{p}^{-m}\right\}=\left\{\beta_{p}^{m}, \beta_{p}^{m-2}, \ldots, \beta_{p}^{2-m}, \beta_{p}^{-m}\right\}
$$

Then that $p \in S$ implies that $\alpha_{p}^{n}=1$ for some nonzero $n$ dependent on $m$. This is because, $a_{f}(p) \neq a_{g}(p)$ implies that $\left\{\alpha_{p}, \alpha_{p}^{-1}\right\} \neq\left\{\beta_{p}, \beta_{p}^{-1}\right\}$. Then $\beta_{p}=\alpha_{p}^{n^{\prime}}\left(m\right.$ is odd and hence $\beta_{p}$ is inside the set) for some $n^{\prime}$ satisfying $1<\left|n^{\prime}\right| \leq m$. However, $\beta_{p}^{m} \in\left\{\alpha_{p}^{m}, \alpha_{p}^{m-2}, \ldots, \alpha_{p}^{2-m}, \alpha_{p}^{-m}\right\}$ and hence $\beta_{p}^{m}=\alpha_{p}^{m^{\prime}}$ for some $m^{\prime}$ satisfying $\left|m^{\prime}\right| \leq m$. This shows

$$
\alpha_{p}^{m^{\prime}}=\beta_{p}^{m}=\alpha_{p}^{m n^{\prime}} .
$$

It is easy to see that $0<\left|m n^{\prime}-m^{\prime}\right| \leq 2 m^{2}$. This will show that $\alpha_{p}^{n}=1$ for some nonzero $n$. In this case, we define

$$
S_{n}=\left\{p \in \mathcal{P} \mid \alpha_{p}^{n}=1\right\}
$$

Then we can find a large $N$ (for example, we can take $N=\left(2 m^{2}\right)$ !) such that $S \subseteq S_{N}$. and hence $1 / 4 \leq \underline{\delta}(S) \leq \underline{\delta}\left(S_{N}\right) \leq \bar{d}\left(S_{N}\right)$.

However, we can show that $\bar{d}\left(S_{N}\right)$ is small by Sato-Tate [BLGHT11]: set $\alpha_{p}=e^{i \theta_{p}}$. (Here I would assume that $\theta_{p} \in[0, \pi]$ since we can replace $\alpha_{p}$ by $\alpha_{p}^{-1}$.) Then the Sato-Tate predicts:

$$
\frac{\#\left\{p \in \mathcal{P} \mid p \leq x, \theta_{p} \in(\alpha, \beta)\right\}}{\#\{p \in \mathcal{P} \mid p \leq x\}} \sim \frac{2}{\pi} \int_{\alpha}^{\beta} \sin ^{2} \theta d \theta
$$

Then set $I_{N}$ to be the set supported on $[0, \pi]$ satisfying the following conditions: (i) let $x \in[0, \pi]$, if $e^{i N x}=1$, then $x \in I_{N}$; (ii) the measure of $I_{N}$ is small such that

$$
\frac{2}{\pi} \int_{I_{N}} \sin ^{2} \theta d \theta<\frac{1}{8}
$$

This will show that

$$
\bar{d}\left(S_{N}\right) \leq \limsup _{x \rightarrow \infty} \frac{\#\left\{p \in \mathcal{P} \mid p \leq x, \theta_{p} \in I_{N}\right\}}{\#\{p \in \mathcal{P} \mid p \leq x\}}=\lim _{x \rightarrow \infty} \frac{\#\left\{p \in \mathcal{P} \mid p \leq x, \theta_{p} \in I_{N}\right\}}{\#\{p \in \mathcal{P} \mid p \leq x\}}<\frac{1}{8}
$$

A contradiction.
When $m$ is even, we define the set

$$
S=\left\{p \mid a_{f}(p)^{2} \neq a_{g}(p)^{2}\right\}=\left\{p \mid a_{\operatorname{Sym}^{2} f}(p) \neq a_{\operatorname{Sym}^{2} g}(p)\right\}
$$

Since $f$ and $g$ are cusp forms of $\mathrm{SL}_{2}(\mathbb{Z}), \operatorname{Sym}^{2} f$ is not isomorphic to $\operatorname{Sym}^{2} g$. It is known, by Lemma 3.1, that $\operatorname{Sym}^{2} f \otimes \operatorname{Sym}^{2} f$ can be written as the isobaric sum of cuspidal representations. Then by [Wal21, Theorem 1.6], we know that

$$
\underline{\delta}(S) \geq \frac{1}{28} .
$$

Fix $m \geq 4$ even and we assume that $\operatorname{Sym}^{m} f$ is isomorphic to $\operatorname{Sym}^{m} g$. Then for each finite $p$, we have:

$$
\left\{\alpha_{p}^{m}, \alpha_{p}^{m-2}, \ldots, \alpha_{p}^{2-m}, \alpha_{p}^{-m}\right\}=\left\{\beta_{p}^{m}, \beta_{p}^{m-2}, \ldots, \beta_{p}^{2-m}, \beta_{p}^{-m}\right\} .
$$

It is clear that $\left\{\alpha_{p}^{2}, \alpha_{p}^{-2}\right\} \neq\left\{\beta_{p}^{2}, \beta_{p}^{-2}\right\}$ by the definition of $S$. Then we consider two separated cases: (i) $\beta_{p}^{2}=1$ and (ii) $\beta_{p}^{2} \neq 1$. For the first case, we know that the right hand of the set is just $\{1, \ldots, 1\}$ and this will give $\alpha_{p}^{2}=1$. For the second case, we assume that $\beta_{p}^{2}=\alpha_{p}^{n^{\prime}}$ with some integer $n^{\prime}$ satisfying $4 \leq\left|n^{\prime}\right| \leq m$. By a similar argument as the odd case, we can find nonzero $n$ such that $\alpha_{p}^{n}=1$ and such $n$ satisfies $|n| \leq 2 m^{2}$. there exists a large integer $N$ such that $S \subseteq S_{N}$. Then a similar argument will show that

$$
\frac{1}{28} \leq \underline{\delta}(S) \leq \bar{d}\left(S_{N}\right)<\frac{1}{56} .
$$

A contradiction.
Combine [CKM04, Theorem 9.2], Lemma 3.2 and [IK04, Theorem 5.13], we have the following results:

Lemma 3.3. Let $f, g$ be normalized holomorphic Hecke eigenforms for $\mathrm{SL}_{2}(\mathbb{Z})$. Then foe any positive integers $m_{1}, m_{2} \geq 1$, we have:

$$
\sum_{p \leq x} \frac{\lambda_{\operatorname{Sym}^{m_{1}} f}(p) \lambda_{\mathrm{Sym}^{m_{2}} g}(p)}{p}=\delta_{m_{1}, m_{2}} \delta_{f, g} \log \log x+O(1) .
$$

Here $\delta_{m, n}=1$ if and only if $m=n$ and it is 0 otherwise. $\delta_{f, g}=1$ if and only if $f=g$ and it is 0 otherwise.

Proof of Corollary 1.4: Let $\pi=\operatorname{Sym}^{m_{1}} f \otimes \operatorname{Sym}^{m_{2}} g$. and hence $L(s, \pi)$ is of degree $\left(m_{1}+1\right)\left(m_{2}+1\right)$. It is known that $L(s, f)$ and $L(s, g)$ satisfy the Ramanujan conjecture and so is $L(s, \pi)$. In addition, we have

$$
\lambda_{\pi}(p)^{2}=\lambda_{\pi \otimes \pi}(p)
$$

and

$$
\lambda_{\pi}(p)^{4}=\lambda_{\pi \otimes \pi \otimes \pi \otimes \pi}(p)
$$

Therefore, by Lemma 3.1 and Lemma 3.3, we have

$$
\sum_{p \leq x} \frac{\lambda_{\pi}(p)^{2}}{p}=\log \log x+O(1)
$$

and

$$
\sum_{p \leq x} \frac{\lambda_{\pi}(p)^{4}}{p}=\left(m_{1}+1\right)\left(m_{2}+1\right) \log \log x+O(1)
$$

Then by Theorem 1.1, we finish the proof.

## 4 Proof of Theorem 1.5

For the Maass form case, we do not have the Ramanujan type bound anymore. However, we can prove the following proposition, which is an analogue of Shiu's result [Shi80, Theorem 1]:

Proposition 4.1. Let $L(s, \pi)$ be a self-dual L-function of degree $d+1$. For simplicity, we assume that $\pi$ is unramified at all finite places. We further assume the following conditions:
(a) There exists a $\delta>0$ such that, for the local parameters $\alpha_{i, p} i=1, \ldots, d+1$, we have $\left|\alpha_{i, p}\right| \leq$ $p^{1 / 2-\delta}$.
(b) The Rankin-Selberg L-function $L(s, \pi \otimes \pi)$ exists and only has a simple pole at $s=1$.

Then we have:

$$
\sum_{n \leq x}\left|\mu_{\pi}(n)\right| \ll \frac{x}{\log x} \exp \left(\sum_{p \leq x} \frac{\left|\mu_{\pi}(p)\right|}{p}\right)
$$

Remark 4.2. Let $\pi$ be a self-dual automorphic cuspidal representation of $\mathrm{GL}_{d+1}\left(\mathbb{A}_{\mathbb{Q}}\right)$. Then Luo, Rudnick and Sarnak [LRS99] showed that $\left|\alpha_{i, p}\right| \leq p^{1 / 2-\delta}$ with $\delta=\frac{1}{(d+1)^{2}+1}$. In addition, the RankinSelberg convolution $L(s, \pi \otimes \pi)$ is defined and it only has a simple pole at $s=1$. (See [CKM04, Theorem 9.2].) In this case, we can apply the proposition to any self-dual automorphic cuspidal representations.

Proof of Proposition 4.1: We will use the following lemma [VKS22, Lemma 3.6]:
Lemma 4.3. Let $g(n)$ be a nonnegative multiplicative function. Suppose that there exists an increasing function $h_{1}(x)$ and a positive function $h_{2}(x)$ such that

$$
\begin{aligned}
\sum_{p \leq x} g(p) \log p & \ll x h_{1}(x) \\
\sum_{p \leq x} \sum_{\alpha \geq 2} \frac{g\left(p^{\alpha}\right)}{p^{\alpha}} \log p^{\alpha} & \ll h_{2}(x) .
\end{aligned}
$$

Then we have:

$$
\sum_{n \leq x} g(n) \ll\left(h_{1}(x)+h_{2}(x)+1\right) \frac{x}{\log x} \sum_{n \leq x} \frac{g(n)}{n} .
$$

Set $g(n)=\left|\mu_{\pi}(n)\right|$ and we will prove

$$
\begin{array}{r}
\sum_{p \leq x}\left|\mu_{\pi}(p)\right| \log p \ll x \\
\sum_{p \leq x} \sum_{\alpha \geq 2} \frac{\left|\mu_{\pi}\left(p^{\alpha}\right)\right|}{p^{\alpha}} \log p^{\alpha} \ll 1 .
\end{array}
$$

(This shows that we can take $h_{1}(x)=1$ and $h_{2}(x)=1$.) For the first inequality, we recall:

$$
-\frac{L^{\prime}(s, \pi \otimes \pi)}{L(s, \pi \otimes \pi)}=\sum_{n=1}^{\infty} \frac{\Lambda_{\pi \otimes \pi}(n)}{n^{s}}
$$

for $\operatorname{Re}(s) \gg 1$. It can be shown that $\Lambda_{\pi \otimes \pi}$ is a nonnegative function supported on prime powers and $\Lambda_{\pi \otimes \pi}(p)=\lambda_{\pi}(p)^{2} \log p=\mu_{\pi}(p)^{2} \log p$. Then by Lemma 5.9 and Theorem 5.13 in [IK04], we have:

$$
\sum_{n \leq x} \Lambda_{\pi \otimes \pi}(n) \sim x
$$

In this case, we can apply Cauchy's inequality:

$$
\sum_{p \leq x}\left|\mu_{\pi}(p)\right| \log p \leq\left(\sum_{p \leq x} \mu_{\pi}(p)^{2} \log p\right)^{1 / 2}\left(\sum_{p \leq x} \log p\right)^{1 / 2} \ll x
$$

On the other hand, we know that $\left|\alpha_{i, p}\right| \leq p^{1 / 2-\delta}$ and hence $\left|\mu_{\pi}\left(p^{\alpha}\right)\right| \ll p^{\alpha(1 / 2-\delta)}$ for all $\alpha \geq 1$. We also have $\left|\mu_{\pi}\left(p^{\alpha}\right)\right|=0$ for $\alpha>m+1$. In this case,

$$
\sum_{p \leq x} \sum_{\alpha \geq 2} \frac{\left|\mu_{\pi}\left(p^{\alpha}\right)\right|}{p^{\alpha}} \log p^{\alpha} \ll \sum_{p \leq x} \sum_{\alpha=2}^{m+1} \frac{1}{p^{\alpha(1 / 2+\delta)}} \log p^{\alpha} \ll 1 .
$$

Combine with Lemma 4.3, and we have:

$$
\sum_{n \leq x}\left|\mu_{\pi}(n)\right| \ll \frac{x}{\log x} \sum_{n \leq x} \frac{\left|\mu_{\pi}(n)\right|}{n}
$$

and we will show:

$$
\sum_{n \leq x} \frac{\left|\mu_{\pi}(n)\right|}{n} \ll \exp \left(\sum_{p \leq x} \frac{\left|\mu_{\pi}(p)\right|}{p}\right) .
$$

Since $\left|\mu_{\pi}(n)\right|$ is a multiplicative function, we have:

$$
\sum_{n \leq x} \frac{\left|\mu_{\pi}(n)\right|}{n} \leq \prod_{p \leq x}\left(1+\frac{\left|\mu_{\pi}(p)\right|}{p}+\cdots+\frac{\left|\mu_{\pi}\left(p^{m+1}\right)\right|}{p^{m+1}}\right) .
$$

Then by $\log (1+x) \leq x$ for $x \geq 0$, we have:

$$
\begin{aligned}
\prod_{p \leq x}\left(1+\frac{\left|\mu_{\pi}(p)\right|}{p}+\cdots+\frac{\left|\mu_{\pi}\left(p^{m+1}\right)\right|}{p^{m+1}}\right) & \leq \exp \left(\sum_{p \leq x} \sum_{\alpha=1}^{m+1} \frac{\left|\mu_{\pi}\left(p^{\alpha}\right)\right|}{p^{\alpha}}\right) \\
& =\exp \left(\sum_{p \leq x} \frac{\left|\mu_{\pi}(p)\right|}{p}+\sum_{p \leq x} \sum_{\alpha=2}^{m+1} \frac{\left|\mu_{\pi}\left(p^{\alpha}\right)\right|}{p^{\alpha}}\right) .
\end{aligned}
$$

We can show that

$$
\sum_{p \leq x} \sum_{\alpha=2}^{m+1} \frac{\left|\mu_{\pi}\left(p^{\alpha}\right)\right|}{p^{\alpha}} \ll 1
$$

since $\left|\mu_{\pi}\left(p^{\alpha}\right)\right| \ll p^{\alpha(1 / 2-\delta)}$. This will finish the proof.
Proof of Theorem 1.5: By Proposition 4.1, it suffices to show:

$$
\sum_{p \leq x} \frac{\left|\lambda_{\phi}(p)\right|}{p} \leq\left(1-\frac{1}{12}\right) \log \log x
$$

We consider the following inequality in [Hol09, Equation (65)]:

$$
t^{1 / 2} \leq 1+\frac{1}{2}(t-1)-\frac{1}{9}(t-1)^{2}+\frac{1}{36}(t-1)^{3}
$$

which is true for $t \geq 0$. Take $t=\left|\lambda_{\phi}(p)\right|^{2}$, and we obtain that

$$
\left|\lambda_{\phi}(p)\right| \leq\left(1-\frac{1}{2}-\frac{1}{9}-\frac{1}{36}\right)+\left(\frac{1}{2}+\frac{2}{9}+\frac{1}{12}\right) \lambda_{\phi}(p)^{2}+\left(-\frac{1}{9}-\frac{1}{12}\right) \lambda_{\phi}(p)^{4}+\frac{1}{36} \lambda_{\phi}(p)^{6}
$$

We have the following relations:

$$
\lambda_{\phi}(p)^{4}=2+3 \lambda_{\operatorname{Sym}^{2} \phi}(p)+\lambda_{\operatorname{Sym}^{4} \phi}(p)
$$

and

$$
\lambda_{\phi}(p)^{6}=5+8 \lambda_{\operatorname{Sym}^{2} \phi}(p)+4 \lambda_{\operatorname{Sym}^{4} \phi}(p)+\lambda_{\operatorname{Sym}^{2} \phi}(p) \lambda_{\operatorname{Sym}^{4} \phi}(p)
$$

By [IK04, Theorem 5.13], we can show that

$$
\sum_{p \leq x} \frac{\left|\lambda_{\phi}(p)\right|^{2}}{p}=\log \log x+O(1)
$$

and

$$
\sum_{p \leq x} \frac{\lambda_{\operatorname{Sym}^{j} \phi}(p)}{p}=O(1)
$$

for $j=2,3,4$. (This is due to the fact that they are cuspidal representations of $\mathrm{GL}_{j+1}$.)
We can also write that $\lambda_{\operatorname{Sym}^{2} \phi}(p) \lambda_{\operatorname{Sym}^{4} \phi}(p)=\lambda_{\operatorname{Sym}^{3} \phi}(p)^{2}-1$. By by Lemma 5.9 and Theorem 5.13 in [IK04], it can be shown that, for $\pi=\operatorname{Sym}^{3} \phi$, one has:

$$
\sum_{n \leq x} \frac{\Lambda_{\pi \otimes \pi}(n)}{n \log n} \sim \log \log x
$$

where $\Lambda_{\pi \otimes \pi}(n)$ is the von Mangoldt function associated to $\pi \otimes \pi$. Therefore, we have:

$$
\sum_{p \leq x} \frac{\lambda_{\operatorname{Sym}^{3} \phi}(p)^{2}}{p} \leq \log \log x
$$

Combining all the results together, we have:

$$
\sum_{p \leq x} \frac{\left|\lambda_{\phi}(p)\right|}{p} \leq\left(1-\frac{1}{12}\right) \log \log x
$$

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